Dimension results for inhomogeneous Moran set constructions

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Abstract

We compute the Hausdorff, upper box and packing dimensions for certain inhomogeneous Moran set constructions. These constructions are beyond the classical theory of iterated function systems, as different nonlinear contraction transformations are applied at each step. Moreover, we also allow the contractions to be weakly conformal and consider situations where the contraction rates have an infimum of zero. In addition, the basic sets of the construction are allowed to have a complicated topology such as having fractal boundaries. Using techniques from thermodynamic formalism we calculate the fractal dimension of the limit set of the construction. As a main application we consider dimension results for stochastic inhomogeneous Moran set constructions, where chaotic dynamical systems are used to control the contraction factors at each step of the construction.

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1 Introduction

A systematic study on the classical theory of iterated function systems (IFS) has been developed in the pioneering work of Moran [14] and Bowen [2], and has been successfully applied in the study of dimension theory (e.g. Bowen's formalism for $C^{1+\epsilon}$ repellers [3, 4, 18]). However, most scenarios require the iterated function system IFS to be *conformal*, and step independent. In this paper we go beyond these classical settings, and consider inhomogeneous Moran set constructions. The main difficulties encountered on estimating the fractal dimension for these constructions are as follows. Firstly, the nonlinear contractions in the IFS are step dependent. Secondly, these contractions are allowed to be weakly conformal (in ways that we will make precise). We also allow the basic sets of the construction to have wild topological properties (such as fractal boundaries), and permit arbitrary placement of the basic sets, subject to these sets being separated. To study inhomogeneous Moran set constructions, we combine various approaches such as those considered in [6, 8, 10] and [1, 3, 16, 17]. Our aim is to form a unified approach in the computation of fractal dimension for such inhomogeneous constructions.

To obtain concrete results on the fractal dimensions such as Hausdorff, upper-box and packing dimension we introduce the main geometrical hypotheses in Section 2. These include assumptions on the degree of nonlinearity permitted on the contractions, and control on the placement of the basic sets in terms of their separation (rather than their precise location). Within this section we also introduce the mechanism of symbolic codings used to describe the basic sets of the construction. In particular, when the IFS is affine or one dimensional cookie-cutter-like, our dimension results on inhomogeneous Moran sets coincide with the results obtained in [6, 8, 10, 21]. In our weakly conformal case, we permit no specific control on the distortion or smoothness of the contraction maps except for continuity. Instead we concentrate on the cardinality of the Moran covering as well as the existence of a *Gibbs-like* measure. We also consider constructions defined on sub-symbolic spaces. In particular, we consider sub-spaces formed by placing restrictions on the sequence of admissible words, for example by introducing a transition matrix. We study the corresponding fractal dimension when the sequence of admissible words is restricted, see Section 2.2. These constructions can be viewed as generalized versions of *graph directed Markov systems* (see [13]).

The main dimension results are presented in Section 3, where we determine the fractal dimension of a limit set F in terms of a sequence of pre-dimensions s_k. The pre-dimension sequence depends on the first k steps of the construction, and for non-linear constructions we take s_k to be the zero of a corresponding pressure equation $P_k(s\Phi_k) = 0$, with a defined potential Φ_k, see Section 2.3. For nonlinear constructions of inhomogeneous Moran sets, our approach extends the theory developed in [17], where they primarily control the geometry using a single vector (of contraction constants) with a finite number of components. In our case we work with a countable sequence of vectors, and the geometry of the construction is controlled using this vector sequence, see Section 2.3. Moreover we consider scenarios where the infimum of the contraction vector components is equal to zero, and comment on situations where the supremum of the contraction vector components equals 1. For example, we believe our techniques will extend to inhomogeneous constructions generated by nonlinear cookie cutters with parabolic fixed points. In the context of IFS having parabolic fixed points, see [7, 19, 12].

As another novelty, we also consider stochastic constructions of inhomogeneous Moran sets and give corresponding dimension results. This is discussed in Section 4. For such constructions, we use a stationary stochastic process to generate the k-step contraction rates, for example by taking a time series of observations on an ergodic transformation (see [20]). This approach appears to be new, at least relative to classical stochastic constructions mentioned in [5, 21]. This gives an alternative approach for constructing random fractals using ergodic and statistical properties of dynamical systems. We study the typical (almost sure) fractal dimension, and further investigations might include studying the largest/smallest dimensions that can arise (e.g. utilizing ideas from ergodic optimization theory [9]). We further consider stochastic constructions where the infimum of the contraction vector components is equal to zero, and where the corresponding supremum equals 1.

The formal proofs of the dimension results are presented in Section 5, with background on dimension theory and thermodynamic formalism presented in Section 6.

2 Geometric and symbolic constructions

2.1 Symbolic spaces for inhomogeneous Moran set constructions

We define the following symbolic space. For a sequence of positive integers $\{n_k\}_{k\geq 1}$ and any $k \in \mathbb{N}$, let

$$D_k = \{(i_1, i_2, \cdots, i_k); \quad 1 \leq i_j \leq n_j, 1 \leq j \leq k\} \quad \text{with} \quad D_0 = \emptyset, \tag{1}$$

and define

$$D = \bigcup_{k=0}^{\infty} D_k. \tag{2}$$

The set D_k contains all words of length k. The collection D is a countable collection of level sets.

Definition 1 *Given a map $f : \mathbb{R}^d \to \mathbb{R}^d$, we define the class $\Im$ such that if $f \in \Im$ then*

(H1) There exists a compact forward invariant set A, such that $f(A) \subset A$;

(H2) For any compact set $B \subset A$, $\mathrm{diam}(f^n(B)) \to 0$ as $n \to \infty$.

We remark that for any $f \in \Im$, then $\bigcap_{n=1}^{\infty} f^n(A)$ is a singleton. If $f_1, f_2 \in \Im$ share the same forward invariant set A, then both $f_2 \circ f_1$ and $f_1 \circ f_2 \in \Im$. We say that a map f is *contracting* if there exists a $0 < c < 1$ such that for all $x, y \in \mathbb{R}^d$, $d(f(x), f(y)) \leq c \cdot d(x, y)$. If f is contracting then $f \in \Im$, but the converse need not be true.

Definition 2 *A family of compact sets is called* **basic sets** $\Omega = \{\triangle_\omega \subset \mathbb{R}^d, \omega \in D\}$, *if this family of sets satisfies:* $\lim_{k\to\infty} \max_{\omega \in D_k} \mathrm{diam}(\triangle_\omega) = 0$.

Based on D and the class $\Im$ of maps, we consider the following *Moran structure conditions* (MSC) for a class of sets $\Omega = \{\triangle_\omega, \omega \in D\}$, where $\triangle_\omega \subset \mathbb{R}^d$ and $\omega = (i_1, i_2, \ldots, i_k)$ is a finite *word* in D. Given words $\omega, \omega' \in D$ we define $\omega * \omega'$ as the concatenation of the two words (when this is still defined in D).

Definition 3 *Given a basic set $\triangle \subset \mathbb{R}^d$ and a sequence of contractions $\{f_{j,i}\Im : i \leq n_j, j \geq 1\}$ we say that $\Omega = \{\triangle_\omega, \omega \in D\}$ satisfies (MSC) with respect to D if the following hold.*

(A1) Suppose $k \geq 1$, $\omega \in D_{k-1}$ and $\omega * j \in D_k$ (for $1 \leq j \leq n_k$). Then elements of $\triangle_{\omega * j}$ are completely determined by elements of $\triangle_\omega$ and the vector of maps $\Xi_k = (f_{k,1}, f_{k,2}, \ldots, f_{k,n_k}) \in \Im$, i.e., $\triangle_{\omega*j} \subseteq \triangle_\omega$ and $\triangle_{\omega*j} = f_{k,j}(\triangle_\omega)$. Moreover if $\omega = (i_1, i_2, \ldots, i_k)$, then we define $f_\omega = f_{1,i_1} \circ f_{2,i_2} \circ \cdots \circ f_{k,i_k}$, so that $f_\omega(\Delta) = \Delta_\omega$.

(A2) The strong separation condition holds: given any k and $\omega, \omega' \in D_k$ with $\omega \neq \omega'$ then

$$\triangle_\omega \cap \triangle_{\omega'} = \emptyset.$$

For a given Ω, we define

$$E_k = \bigcup_{\omega \in D_k} \triangle_\omega, \ F = \bigcap_{k \in \mathbb{N}} E_k. \tag{3}$$

The set F is a compact set, and by the strong separation condition (A2) is totally disconnected. So far we have made no assumptions on the topology of the basic set $\triangle$, nor on the sets $\triangle_\omega$ ($\omega \in D$) other than these sets being compact. In particular they need not to be connected, and their boundaries could be fractal. It is sufficient for our purposes to work with a weaker version of (A2), and we say that the *weak separation condition holds* if

(A2') For any $\omega, \omega' \in D$ with $\omega \neq \omega'$:

$$\{\triangle_\omega \cap \triangle_{\omega'}\} \cap F = \emptyset.$$

Definition 4 *Given F as in equation* (3)*, we call F a* **generalized Moran set** *(GMS) if F satisfies (A1) and (A2').*

See Fig 1 for the geometrical interpretation.

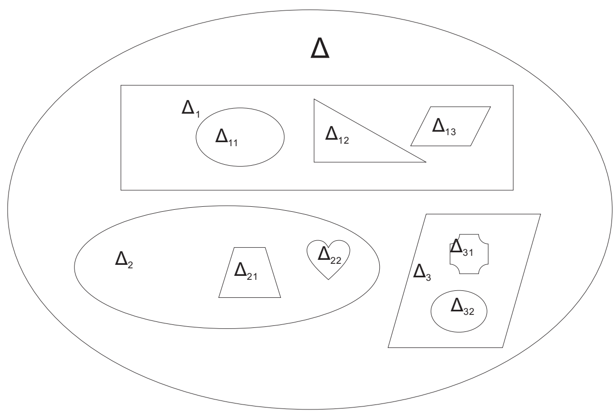

Figure 1: Schematic representation of the geometric construction of a general Moran set F.

Now define the set

$$D^* = \{(i_1, i_2, \cdots, i_k, \cdots) : 1 \leq i_j \leq n_j, \ j \geq 1\}.$$

This set consists of infinite strings, and any $\omega \in D^*$ has the representation $\omega = (i_1, i_2, \ldots)$. Given $\omega \in D^*$, we write $C_{(i_1, \ldots i_k)}(\omega) \subset D^*$ as the k-length cylinder set. Given D^* and F, there is a canonical projection map $\mathcal{X} : D^* \to F$ which assigns to each $\omega = (i_n)_{n=1}^\infty$ the point $x \in F$ given by $\bigcap_k \triangle_{(i_1, \ldots i_k)}$.

We can turn D^* into a metric space by assigning the distance function $d(\omega, \omega')$ to points $\omega', \ \omega \in D^*$ as follows:

$$n(\omega, \omega') := \min\{i | \omega_j = \omega'_j \ \text{ for } 0 < j < i \text{ but } \omega_i \neq \omega'_i\}, \ \text{if } \omega \neq \omega',$$

and $n(\omega,\omega) := \infty$. For given $p_i < n_i^{-1}$ we set $d(\omega,\omega') = \prod_{i=1}^{n(\omega,\omega')} p_i$. Then (D^*, d) is a compact metric space. It is easy to see that D^* is a generalization of traditional symbolic space, since if $n_k = p$ is a constant, then $D^* = \Sigma_p^+$, where $\Sigma_p^+ = \{1,\ldots,p\}^{\mathbb{N}}$. When n_k is not constant the shift map σ on D^* does not preserve D^* in general. We consider a sequence of symbolic spaces that can be thought as approximations to D^*. These symbol spaces are generated from the sets D_k.

Definition 5 *Given $D = \bigcup_k D_k$, the symbol space $[D_k]$ is defined as the set of infinite strings, with indices corresponding to elements of D_k. That is*

$$[D_k] = \{\mathbf{w} = (\omega_i)_{i=1}^{\infty} = (\omega_1, \omega_2\ \ldots,), \, \omega_i \in D_k\}.$$

The associated shift map $\sigma_k : [D_k] \to [D_k]$ is defined by:

$$\sigma_k(\omega_1, \omega_2\ \ldots,) = (\omega_2\, \omega_3, \ldots,), \quad \textit{with } (\omega_i)_{i=1}^{\infty} \in [D_k].$$

Notice that $[D_k]$ is isomorphic to $\Sigma_{p_k}^+$ with $p_k = \mathrm{card}(D_k)$.

2.2 Sub-spaces of symbolic constructions

So far we have considered all admissible collections of words in $D = \cup_k D_k$. Instead, we can consider subsets of words $Q_k \subset D_k$, with $Q = \cup_k Q_k \subset D$. If $\{A^{(k)}\}$ is a sequence of (transition) matrices, having entries in $\{0,1\}$ then admissible words in Q may be characterized in terms of products of these matrices. In particular we can write

$$Q_k = \{\omega = (i_1,\ldots,i_k) \in D_k : \, A^{(1)}_{i_1 i_2} A^{(2)}_{i_2 i_3} \cdots A^{(k-1)}_{i_{k-1} i_k} = 1\}, \quad Q = \bigcup_k Q_k. \tag{4}$$

Thus with Q and Q_k in place of D, resp. D_k, we can produce constructions in analogy to those considered in Definitions 3 and 4, but now for the class of sets $\Omega(Q) = \{\Delta_\omega \subset \mathbb{R}^d, \omega \in Q\}$. The corresponding limit set F defined by

$$E_k = \bigcup_{\omega \in Q_k} \triangle_\omega, \quad F = \bigcap_{k \in \mathbb{N}} E_k, \tag{5}$$

will be referred to as a generalized Moran set associated to Q. We define

$$Q^* = \{(i_1, i_2, \cdots, i_k, \cdots) : \, A^{(k)}_{i_k i_{k+1}} = 1, k \geq 1\},$$

and given $\omega \in Q^*$, we write $C_{(i_1,\ldots i_k)}(\omega) \subset Q^*$ as the k-length cylinder set. There is again a canonical projection map $\mathcal{X} : D^* \to F$ that takes $\omega = (i_n)_{n=1}^{\infty}$ to the $x \in F$ given by $\bigcap_k \triangle_{(i_1,\ldots i_k)}$. We again can turn Q^* into a metric space (using the metric inherited from that of D^*), and we define the symbol space $[Q_k]$ in direct analogy to $[D_k]$.

Since Q can be quite general, we will mainly consider the case where the transition matrices $A^{(k)} := A$ are fixed $p \times p$ matrices (and hence $n_k = p$ for each k). We can then find the fractal dimension of F in terms of the (spectral) properties of A, and in terms of the contraction vector sequence Ξ_k as defined in condition (A1).

2.3 Conformal constructions and constructions bounded via upper/lower estimating vectors

To obtain explicit estimates on the Hausdorff dimension of F, some restrictions on the basic sets $\triangle_\omega$ are required. In particular we require control on the diameter of $\triangle_\omega$ with respect to the level set D_k that ω belongs to. In particular we require that their diameters shrink exponentially fast with k. We also require control the geometry of $\triangle_\omega$ via a technical condition restricting the number of $\triangle_\omega$ (of a certain size-scale) that can intersect with a given ball $B(x,r)\in\mathbb{R}^d$ where $x\in F$. For self similar constructions, control on the geometry is specified in [17] by use of lower, and upper estimating vectors. We adapt these methods for the non-self similar constructions. Let $\overline{\Psi}=\{\overline{\Psi}^{(k)},\,k\in\mathbb{N}\}$ denote a countable collection of vectors $\overline{\Psi}^{(k)}$ with

$$\overline{\Psi}^{(k)}=(\Psi^{(k)}(\omega))_{\omega\in D_k}.$$

Here ω has the representation as some $(i_1,\ldots,i_k)\in D_k$. Given $\omega\in D_k$, we assume that there is a sequence of constants $c_{i_1},\ldots c_{i_k}$ such that

$$\Psi^{(k)}(\omega)=c^{(1)}_{i_1}c^{(2)}_{i_2}\ldots c^{(k)}_{i_k}=\prod_{j=1}^{k}c^{(j)}_{i_j}.$$

For notational simplicity we sometimes write $\Psi^{(k)}_\omega:=\Psi^{(k)}(\omega)$. In relation to the sequence $\overline{\Psi}^{(k)}$ we define $\tilde{\Xi}_k$ to be the k-step vector sequence:

$$\tilde{\Xi}_k=(c^{(k)}_1,c^{(k)}_2,\ldots,c^{(k)}_{n_k}).$$

For example, if the k-step vector $\Xi_k=(f_{k,i})_{i=1}^{n_k}$ consists of affine maps, each with contraction rate $c^{(k)}_{i_k}$, then a natural choice for $\tilde{\Xi}_k$ would be the vector sequence of corresponding contraction ratios.

Definition 6 (Basic vectors) *The collection of vectors $\overline{\Psi}=\{\overline{\Psi}^{(k)}\}_{k=1}^{\infty}$ is called a* **basic collection** *of vectors if for all $k\geq 1$ and all $\omega=(i_1,\ldots,i_k)\in D_k$, the sequence $\Psi^{(k)}(\omega)$ satisfies*

$$\sup_{k\in\mathbb{N},1\leq j\leq n_k} c^{(k)}_j<1. \tag{6}$$

Definition 7 (UE vectors) *A basic collection of vectors $\overline{\Psi}=\{\overline{\Psi}^{(k)}\}_{k=1}^{\infty}$ is called an* **upper estimating** *(UE) collection of vectors if for any k and $\omega\in D_k$:*

$$\mathrm{diam}(\triangle_\omega)\leq C\Psi^{(k)}_\omega,$$

and the constant $C>0$ is independent of ω and k.

To get bounds on the Hausdorff dimension we require further control of the geometry of each Δ_ω. We introduce two definitions: the first is that of conformality, while the second introduces the notion of lower-estimating vectors for a geometric construction.

Definition 8 (Conformal vectors) *Given a basic collection of vectors* $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}_{k=1}^{\infty}$, *we say that a symbolic construction* $\{\triangle_\omega\}$ *is* **conformal** *(w.r.t.* $\overline{\Psi}$*) if* $\exists\, C > 0$, *such that for each* $k \geq 1$, $\omega \in D_k$, $\exists x \in \Delta_\omega$:

$$B\left(x, \frac{1}{C}\Psi_\omega^{(k)}\right) \subset \triangle_\omega \subset B\left(x, C\Psi_\omega^{(k)}\right). \tag{7}$$

The following geometric constraint is formulated in terms of Moran coverings which we describe as follows, see also [17]. Given a set F, and for any $x \in F$, choose the $\omega \in D^*$ for which $\mathcal{X}(\omega) = x$. By the separation condition, ω is unique. Suppose $0 < r < 1$ is fixed and let $\overline{\Psi}$ be a basic sequence of vectors. Let $n(x)$ be the unique positive integer of the such that

$$\Psi_\omega^{(n(x))} > r \text{ and } \Psi_\omega^{(n(x)+1)} \leq r.$$

If $C(\omega)$ is the corresponding $n(x)$-length cylinder set, we write $\triangle(x) := \mathcal{X}(C(\omega))$. For $x, y \in F$, either $\triangle(x) = \triangle(y)$ or $\triangle(x) \cap \triangle(y) = \emptyset$. The corresponding (disjoint) collection of sets we denote by $\{\triangle^{(j)}\}$, where $F \subset \cup_j \triangle^{(j)}$, and this forms the *Moran covering* of the set F.

Consider the open ball $B(x, r)$ of the radius r centered at the point $x \in F$, and let $N(x, r)$ denote the cardinality of the subset of $\{\triangle^{(j)}\}$ that have non-empty intersection with $B(x, r)$. We have the following definition.

Definition 9 (LE vectors) *If there exists a constant* M *such that the above* $N(x, r) < M$ *for all* $x \in F$, *then we say the collection of vectors* $\overline{\Psi}$ *is* **lower estimating** *(LE).*

In the special case where the vector $\overline{\Psi}$ has the property that $\Psi_\omega^k = \Psi_{\omega'}^{(k)}$ for all $\omega, \omega' \in D_k$, then we call the construction *homogeneous* if such a vector is both (UE) and (LE). The corresponding limit set F is called homogeneous, otherwise in all other cases the construction (and limit set) is *inhomogeneous.*

Pre-dimension sequences

For MSCs arising from non-linear constructions, we determine the dimension of the Moran set F from a sequence of pre-dimensions s_k. These s_k will be prescribed to be the zeros of a functional equation involving the topological pressure. We make this precise as follows. Consider a sequence of pressure functions P_k (for $k \in \mathbb{N}$), and a sequence of potentials Φ_k defined as follows. Suppose that $\overline{\Psi}$ is prescribed, and consider the symbolic space $[D_k]$ together with the shift map $\sigma_k : [D_k] \to [D_k]$. For $\mathbf{w} = (\omega_1, \omega_2, \ldots) \in [D_k]$ let $\Phi_{k,s}(\mathbf{w}) := s \log \Psi^{(k)}(\omega_1)$. This function can be extended to a function on F_k via $\Phi_{k,s}(x) = s \log \Psi^{(k)}(\omega_1)$, where $\mathcal{X}(\mathbf{w}) = x$. We define the corresponding pressure function $P_k : \mathrm{Lip}(F_k) \to \mathbb{R}$ by

$$P_k(\Phi_{k,s}) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{(\omega_1,\ldots,\omega_n)} \inf_{x \in \Delta_{(\omega_1,\ldots,\omega_n)}} \exp\left\{S_n(\Phi_{k,s}(x))\right\} \right), \tag{8}$$

where $\Delta_{\omega_1,\dots,\omega_n} = f_{\omega_1,\dots,\omega_n}(\Delta)$, and $\omega_i \in D_k$. Now we consider the sequence s_k, where s_k is the value of s which solves $P_k(\Phi_{k,s}) = 0$. In particular we consider the (lim)-inf and (lim)-sup of this sequence. We define:

$$s^* := \limsup s_k, \quad \text{and} \quad s_* := \liminf s_k. \tag{9}$$

The main focus of this paper is to consider when s^* is the upper-box dimension of F, and when s_* is the Hausdorff dimension of F.

To obtain dimension estimates for F in terms of zeros of the pressure function we need to assume the existence of a *Gibbs-like* measure on F as follows:

(A3) Given $\beta > 0$, there exists a measure m_Ψ supported on F, and $L > 0$ such that for all $k \geq 1, \omega \in D_k$,

$$\frac{L^{-1}}{\sum_{\omega' \in D_k} (\Psi^{(k)}(\omega'))^\beta} \leq \frac{m_\Psi(\Delta_\omega)}{(\Psi^{(k)}(\omega))^\beta} \leq \frac{L}{\sum_{\omega' \in D_k} (\Psi^{(k)}(\omega'))^\beta} \tag{10}$$

For a range of applications hypothesis (A3) can be verified. For IFS defined by expanding maps, then (A3) typically follows from bounded distortion estimates, see Section 4. Without (A3), assumptions (A1) and (A2) will not ensure that $\dim_H(F) = s_*$.

3 Statement of main results

For general Moran set constructions we now compute (or estimate bounds) on the Hausdorff, upper-box and packing dimensions based on the existence of a countable sequence $\overline{\Psi}$ of upper and lower estimating vectors. We will assume geometrical assumptions (A1), (A2) and existence of a Gibbs-like measure (A3). Applications fitting these geometrical models will be discussed in Section 4. The constant c_* will also be of importance, where we define

$$c_* := \inf_{k \in \mathbb{N}, 1 \leq j \leq n_k} c_j^{(k)}. \tag{11}$$

We will distinguish between cases where $c_* > 0$ and $c_* = 0$.

Theorem 1 *Consider a MSC with F a GMS. Suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}, k \in \mathbb{N}\}$ is a basic sequence of vectors which satisfy the (UE), (LE) properties, and suppose that there exists a Gibbs-like measure m_Ψ satisfying (A3). Assume further that $c_* > 0$, where c_* is defined in equation* (11). *Then*

1. $\dim_H F = \dim_H m_\Psi = s_*$.
2. $\dim_P F$, $\overline{\dim}_B F \leq s^*$.

If instead F satisfies the conformality condition, as in Definition 8 then

$$\dim_P F = \overline{\dim}_B F = s^*.$$

We remark that under assumption (11), the existence of a conformal vector implies the (LE) property; see the proof of Lemma 9. However, the converse does not hold. Under the assumption of a vector being lower estimating, and the construction non-conformal then we only obtain the inequality $\overline{\dim}_B F \leq s^*$. So far, we do not have an explicit construction of a set F for which the inequality is strict.

Suppose now that $c_* = 0$. Then we have to impose conditions on how fast the $\Psi_\omega^{(k)}$ decay to get corresponding results as stated in Theorem 1. For fixed k, we denote

$$M_k := \max_{\omega \in D_k} \Psi_\omega^{(k)}, \ d_k := \min_{1 \leq j \leq n_{k+1}} c_j^{(k+1)}. \tag{12}$$

We have the following result

Theorem 2 *Consider a MSC with F a GMS. Suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}, k \in \mathbb{N}\}$ is a sequence of vectors which satisfy the (UE), (LE) properties, and suppose that there exists a Gibbs-like measure m_Ψ satisfying (A3). Furthermore assume that $c_* = 0$, and*

$$\lim_{k \to \infty} \frac{\log d_k}{\log M_k} = 0, \tag{13}$$

then $\dim_H F = \dim_H(m_\Psi) = s_*$ *and* $\dim_P F$, $\overline{\dim}_B F \leq s^*$, *where d_k, M_k are defined in* (12)*. If instead F satisfies the conformality condition, as in Definition 8 then*

$$\dim_P F = \overline{\dim}_B F = s^*.$$

It is possible to impose alternative conditions on the vectors $\Psi_\omega^{(k)}$ where (11) holds. We consider the following conditions, suppose

$$M = \sup_{k \geq 1} n_k < \infty \tag{14}$$

$$0 < \inf_k \max_{1 \leq j \leq n_k} c_j^{(k)} \leq \sup_k \max_{1 \leq j \leq n_k} c_j^{(k)} < 1. \tag{15}$$

For example, equation (13) can be satisfied for a homogeneous construction having $c_j^{(k)} = c_k$, for $1 \leq j \leq n_k$, and $\inf_k c_j = 0$. However for this example, equation (15) will fail. An example that satisfies (15), but not (13) would be a construction with vector $\tilde{\Xi}_k = (1/4, 1/4, (1/4)^k)$. The following theorem holds.

Theorem 3 *Consider a MSC with F a GMS. Suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}, k \in \mathbb{N}\}$ is a* basic *sequence of vectors which satisfy the (UE), (LE) properties, and suppose that there exists a Gibbs-like measure m_Ψ satisfying (A3). Moreover, suppose that equations* (14), (15) *hold with $c_* = 0$. Then* $\dim_H F = \dim_H(m_\Psi) = s_*$*. If instead, $\overline{\Psi}$ is a* conformal vector *then* $\dim_P F = \overline{\dim}_B F = s^*$.

When $\sup_{k,j} c_j^{(k)} = 1$ and/or when $\sup_k n_k = \infty$ then it is possible to give constructions where $\dim_H(F) \neq \liminf s_k$, and/or $\overline{\dim}_B(F) \neq \limsup s_k$, see [8]. For Moran set constructions modelled by subsets of symbolic spaces then corresponding results hold. We state the following corollary (whose proof follows step by step from the proofs of Theorems 1, 2 and 3).

Corollary 1 *Suppose that F is a GMS generated by a sub-symbolic space $Q_k \subset D_k$, with $n_k = p$ fixed, and allowed words modelled by a (fixed) transition matrix A. Relative to the space Q, suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a sequence of vectors which satisfy the (UE), (LE) properties. Furthermore suppose that relative to the space Q there exists a Gibbs-like measure m_Ψ satisfying (A3). Then the conclusions of Theorems 1, 2 and 3 remain valid.*

4 Applications

We consider applications of Theorems 1, 2 and 3 to a range of examples. We first consider step dependent IFS, and then explore Moran set constructions with stochastic vectors.

4.1 Iterated function systems

In this section we consider IFS defined by sequences of expanding maps.

Suppose that we are given a basic set $\triangle \subset \mathbb{R}^d$ and $\Omega = \{\triangle_w \in \mathbb{R}^d : \omega \in D\}$ satisfies the conditions of MSC as stated in Definition 3. Based on these geometrical constructions, we consider a family of maps $\{T_{i,j}\}$ defined in the following way.

$T_{j,i_j} : \triangle_{i_1,\ldots,i_j} \to \triangle_{i_1,\ldots,i_{j-1}}, \forall i_j = 1, \cdots, n_k$ satisfies the following assumptions:

(IFS1): For $j \geq 1$ and $1 \leq i_j \leq n_j$, $T_{j,i_j} : \triangle_{i_1,\ldots,i_j,} \to \triangle_{i_1,\ldots,i_{j-1}}$ is a full-branch $C^{1+\alpha}$ diffeomorphism. In particular $T_{j,i_j}(\triangle_{i_1,\ldots,i_j,}) = \triangle_{i_1,\ldots,i_{j-1}}, \forall i_j = 1, \cdots, n_k$, and the derivative DT_{j,i_j} is $\alpha-$ Hölder continuous, i.e., there exists a constant $C := C_{j,i_j}$ such that $||DT_{j,i_j}(x) - DT_{j,i_j}(y)|| \leq C||x-y||^\alpha$.

(IFS2): There exists a $\beta := \beta_{j,i_j} > 1$ such that $||T_{j,i_j}(x) - T_{j,i_j}(y)|| \geq \beta||x-y||$, $\forall x, y \in \triangle_{i_1,\ldots,i_j}$;

We take $\Xi_k = (f_{k,1}, \ldots, f_{k,n_k})$ to be the vector of contractions associated to the inverse branches of $(T_{k,1}, \ldots, T_{k,n_k})$ at the $k-$th step. For $\omega = (i_1, \ldots, i_k) \in D_k$ we have $\triangle_\omega = f_\omega(\triangle)$ where $f_\omega = f_{1,i_1} \circ \cdots, \circ f_{k,i_k}$. We state the following result

Corollary 2 *For a family of expanding diffeomorphisms $\{T_i\}$, let $\{\Xi_k\}_{k=1}^\infty$ be the vector sequence of contractions associated to the inverse branches. Consider a GMS, F associated to this $\{\Xi_k\}$. Assume that the $C^{1+\alpha}$ (distance)-expansivity of $\{\Xi_k\}_{k=1}^\infty$ is uniformly bounded, i.e., the sequence $\{\beta_{j,i_j}\}$ is uniformly bounded away from 1, the sequence $\{\det(Df_{j,i_j})\}$ is uniformly bounded away from zero, and the sequence of Hölder constants $\{C_{j,i_j}\}$ is uniformly bounded. Then*

$$\dim_H F = s_*, \quad \dim_P F = \overline{\dim_B} F = s^*,$$

where s_* and s^* are defined in equation (9).

Before giving the proof consider the example where f_i are similarity contractions and the basic sets $\triangle_\omega$ as intervals (or balls) in $\mathbb{R}^d$, see [8]. We show how the corresponding

dimension estimates are obtained by assuming (A1), (A2), and checking (A3). The problem can be reduced to taking a sequence of vectors $\tilde{\Xi}_k$ (associated to Ξ_k) given by

$$\tilde{\Xi}_k = (c_1^{(k)}, c_2^{(k)}, \ldots, c_{n_k}^{(k)}), \tag{16}$$

where the $c_i^{(k)} = Df_{k,i} \mid \Delta$ are positive constants. Assuming (A1) and (A2) there is a similarity transformation f_ω taking Δ to Δ_ω. Moreover, suppose $k \geq 1$, $\omega \in D_{k-1}$ and $\omega * j \in D_k$ (for $1 \leq j \leq n_k$). Then $\triangle_{\omega * j} \subset \triangle_\omega$, and

$$\frac{|\Delta_{\omega * j}|}{|\Delta_\omega|} = c_j^{(k)}.$$

The corresponding pre-dimension sequences $\{s_k\}$ satisfy the equations

$$\prod_{i=1}^{k} \sum_{j=1}^{n_j} (c_j^{(i)})^{s_k} = \sum_{\omega \in D_k} (\mathrm{diam} f_\omega(\Delta))^{s_k} = 1. \tag{17}$$

These equations are equivalent to solving $P(s_k \Phi_k(x)) = 0$, where $\Phi_k(x) = s_k \log \Psi^{(k)}(\mathbf{w}^{(1)})$, $\forall x \in \Delta_{\omega_1, \ldots, \omega_n}$, and $P(\cdot)$ is defined in equation (8). The corresponding Gibb-like measure m_Ψ can be made taken as the weak limit of the sequence of measures m_k, where each m_k is defined on $\omega \in D_\ell, \ell \leq k$ as follows:

$$m_k(\Delta_\omega) = \sum_{i_{\ell+1}, \ldots, i_k} \frac{(c_{1,i_1} c_{2,i_2} \ldots c_{k,i_k})^\beta}{\prod_{j=1}^k \sum_{i=1}^{n_k} c_{j,i}^\beta}, \quad \omega = (i_1, \ldots i_\ell).$$

By linearity of the construction we have $m_k(\Delta_\omega) = m_\ell(\Delta_\omega)$. The estimates are uniform in k, and hence (A3) holds when taking a weak limit of $\{m_k\}$. We therefore obtain by Theorem 1

$$\dim_H(F) = s_*, \quad \overline{\dim_B}(F) = \dim_P(E) = s^*, \tag{18}$$

where

$$s_* = \liminf_{k \to \infty} s_k, \quad s^* = \limsup_{k \to \infty} s_k. \tag{19}$$

Proof of Corollary 2: The key calculation in the nonlinear setting is to use bounded distortion. We show that the construction can be modelled by a basic and conformal vector sequence $\overline{\Psi}$. Furthermore we check that (A3) holds.

First of all, we claim that there exists $D > 0$, independent of k such that for all $x, y \in \triangle$ and $\omega \in D_k$

$$\frac{1}{D} \leq \frac{|\det(Df_\omega(x))|}{|\det(Df_\omega(y))|} \leq D. \tag{20}$$

The proof of the distortion result is based on the chain rule, for the same iterated function system at each level; see [5, 18]. More precisely, we have:

$$
\begin{aligned}
&|\log|\det Df_\omega(x)| - \log|\det Df_\omega(y)||\\
=&\sum_{j=1}^{k}\left|\log|\det Df_{j,i_j}(f_{\omega|j}(x))| - \log|\det Df_{j,i_j}(f_{\omega|j}(y))|\right|\\
\leq&\sum_{j=1}^{k} C_1\left|\det D_{j,i_j}(f_{\omega|j}(x)) - \det D_{j,i_j}(f_{\omega|j}(y))\right|\\
\leq&\sum_{j=1}^{k} C_2||Df_{j,i_j}(f_{\omega|j})(x) - Df_{j,i_j}(f_{\omega|j})(y)||\\
\leq&\sum_{j=1}^{k} C_3||f_{\omega|j}(x) - f_{\omega|j}(y)||^\alpha\\
\leq& C_3\sum_{j=1}^{k}\beta^{-j\alpha}||x-y||^\alpha \leq \frac{C_3\beta^{-\alpha}}{1-\beta^{-\alpha}}||x-y||^\alpha,
\end{aligned}
$$

where for $j \leq k$, $\omega = (i_1, \ldots, i_k) \mid j$ corresponds to the word $(1_1, \ldots, i_j)$. Due to the uniform bounded distortion, these constants $C_i, i = 1, 2, 3$ and β are independent of the choice of k, which implies (20).

From this bounded distortion property (20), we can directly construct a collection of vectors $\overline{\Psi}$ and verify the conformality and (A2). More precisely, for any fixed $x \in \triangle$, let $\Psi_\omega = \sup_{x\in\triangle_\omega}|\det Df_\omega(x)|, \forall\omega = \omega \in D_k$. Then, for all $y \neq x \in \triangle$, we have

$$\frac{1}{D} \leq \frac{|\det Df_\omega(y)|}{\Psi_\omega^{(k)}} \leq D.$$

Thus by the expanding and distortion properties of $\{T_i\}$, the vector sequence $\overline{\Psi}$ is basic and conformal. To check assumption $(A3)$ we take m_Ψ as weak limit of measures m_k, where each m_k is defined on $\omega \in D_\ell, \ell \leq k$ as follows:

$$m_k(\Delta_\omega) = \frac{(\mathrm{diam}(\triangle_\omega))^\beta}{\sum_{\omega\in D_k}(\mathrm{diam}(\triangle_\omega))^\beta}.$$

This is in complete analogy to the linear construction considered for similarity transformations. A computation using bounded distortion, see [10, Prop 2.7], implies that m_Ψ satisfies (A3). The corresponding results on the dimension follow from Theorem 1. $\square$

Corollary 2 extends the results of [10] to higher dimensions, and to situations where the basic sets have fractal boundaries. The results also apply when taking instead complex conformal holomorphic expanding maps on the Riemann sphere $\overline{\mathbb{C}}$. In this case we let $\Psi_\omega^{(k)} := \max_{x\in\Delta_\omega}|\operatorname{Arg}(f'_\omega(x))|$).

So far we have assumed the sequence of vectors to be basic. The authors conjecture that this assumption can be relaxed, and the results extend to the scenario where

the class of maps $\{T_i\}$ are *non-uniformly expanding.* An example would include the parabolic-fixed point family of maps $T_i : [0,1] \to \mathbb{R}$, $\alpha_i \in (0,1)$ given by

$$T_i(x) = \begin{cases} x(1+3x^{\alpha_i}) & \text{if}\, x \in [0,1/2], \\ 3(1-x) & \text{if}\, x \in (1/2,1]. \end{cases} \tag{21}$$

For $\alpha = \alpha_i$ fixed, and potential $\phi(x) = s\log T'(x)$ the corresponding pressure function is no longer analytic in s. There is a critical value $s = s_c$ for which the pressure function undergoes a phase transition (corresponding to derivative singularity). For all $s > s_c$, the pressure function is zero. However it can be shown that $\dim_H(F) = s_c = \inf\{s : P(s\phi) = 0\}$, see [7, 19, 12]. For inhomogeneous Moran set constructions generated by a sequence of maps T_i. The authors conjecture that for a sequence of maps T_i, each having a parabolic fixed point (with parabolic index α_i) the corresponding dimension is given by $\dim_H(F) = s_*$, with $s_* = \liminf_k s_k$, and $s_k = \inf\{s : P_k(s\Phi_k) = 0\}$.

4.2 Stochastic Moran set constructions

In this section we consider Moran set constructions based on stochastic vector models. Given $\omega = (i_1, \ldots, i_k) \in D_k$, we assume the constants $c^{(j)}_{i_j}(\omega)$ that constitute the vector $\overline{\Psi}^{(k)}$ are generated by a stationary stochastic process, such as an ergodic transformation.

Homogeneous-stochastic Moran set constructions

The homogeneous construction is perhaps the simplest example of a MSC. A natural exploration is to consider ways of generating the limit set F via stochastic sequences of contractions. For example, we consider the vector $\overline{\Psi}$ generated stochastically via chaotic maps in the following sense: Let (T, M, μ) be a measure preserving system, where $T : M \to M$ is a map preserving an ergodic measure μ. Given a test function (observable) $\phi : M \to [0,1]$ and initial condition $x \in M$, we let $\Psi^{(k)}_\omega = \prod_{j=1}^k \phi(T^j(x))$ for any $\omega \in D_k$. We assume that $n_k = q$ is fixed, and the conditions of Definition 3 apply. In this case the vector Ξ_k consists of q components each with value $\phi(T^k(x))$. Thus the limit set F (and hence its dimension) depends on the initial value $x \in M$. In this section we primarily investigate the Hausdorff dimension of F, and it's dependency on x. The results are obtained by using methods in ergodic theory.

Theorem 4 *Suppose that (T, M, μ) is an ergodic system, and suppose that $\phi : M \to [0,1)$ is such that $\log\phi \in L^1(\mu)$ with $\int \log\phi < 0$. Suppose further that F is the homogeneous GMS arising from a MSC with a basic vector $\Psi^{(k)}_\omega = \prod_{j=1}^k \phi(T^j(x))$ that is both (LE) and (UE). Assuming (A3), then for μ-a.e. $x \in M$*

$$\dim_H(F) = \dim_P(F) = \overline{\dim_B}(F) = \frac{-\log q}{\int \log\phi d\mu}.$$

Proof: When $\inf_{x\in M}\phi(x) > 0$, Theorem 1 implies that

$$\dim_H(F) = s_*,$$

where

$$s_* = \liminf_{k\to\infty} \left(\frac{\log\left(\prod_{j=1}^{k} \phi \circ T^j(x)\right)}{-k \log q} \right)^{-1}.$$

The Birkhoff Ergodic Theorem implies that μ-a.e. $x \in M$:

$$\lim_{k\to\infty} \frac{1}{k} \log \left(\prod_{j=1}^{k} \phi \circ T^j(x) \right) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \log \phi \circ T^j(x) = \int \log \phi d\mu.$$

Now consider the case where $\inf_{x\in M} \phi(x) = 0$. Since $\inf_{k\in\mathbb{N}, 1\le j\le n_k} c_j^{(k)} = 0$ we need to show that equation (13) applies for μ-typical orbits, and then we apply Theorem 2.

Using the notation of equation (13) we have $d_k = \phi(T^{k+1}(x))$ and $M_k = \prod_{i=1}^{k} \phi(T^i(x))$. Hence,

$$\frac{\log d_k}{\log M_k} = \frac{\log \phi(T^{k+1}(x))}{\sum_{i=1}^{k} \log \phi(T^i(x))} = \frac{k^{-1} \log \phi(T^{k+1}(x))}{k^{-1} \sum_{i=1}^{k} \log \phi(T^i(x))}. \tag{22}$$

Again, by the ergodic theorem, $k^{-1} \sum_{i=1}^{k} \log \phi(T^i(x)) \to \int \log \phi \, d\mu \neq 0$. To show $k^{-1} \log \phi(T^{k+1}(x)) \to 0$ (for μ-a.e. $x \in M$), let $a_k = k$, $b_k = \log(\phi(T^k(x))$ and $S_k := \frac{1}{a_k} \sum_{j=1}^{k} b_k$. Then

$$\frac{a_{k+1}}{a_k} S_{k+1} - S_k = \frac{b_{k+1}}{a_k},$$

and taking limits on both sides implies $\lim_{k\to\infty} b_{k+1}/a_k = 0$. Hence $\log d_k / \log M_k \to 0$ for μ-a.e. $x \in M$, proving the result.

$\square$

Inhomogeneous-stochastic Moran set constructions

Consider a family of maps $\{(T_i, M, \mu_i)\}_{i=1}^{q}$ with $T_i : M \to M$ (M compact), and each T_i preserves an ergodic measure μ_i with density in L^p for some $p > 1$. Given $\mathbf{x} \in M^q$, we can generate a limit set F via a MSC in the following way. Take continuous functions $\phi_i : M \to [0, 1]$, and suppose that the basic vector $\Psi_\omega^{(k)}$ has the form:

$$\Psi_\omega^{(k)} = \prod_{j=1}^{k} \phi_{i_j}(T_{i_j}^{j}(x_{i_j})) : \ \omega = (i_1, \ldots, i_k), \ \mathbf{x} = (x_1, \ldots, x_q). \tag{23}$$

We have the following result.

Theorem 5 *Suppose that $\{T_i, M, \mu_i\}_{i=1}^{q}$ form a mixing system (i.e., each measure μ_i is mixing w.r.t. T_i) and each $\phi_i : M \to [0, 1)$ is positive Hölder continuous with $\int \log \phi_i \, d\mu_i < 0$. Suppose that F is a GMS arising from a MSC with a basic vector $\overline{\Psi}$ generated via the vectors $\tilde{\Xi}_k = (\phi_1(T_1^k(x_1)), \ldots, \phi_t(T_t^k(x_t)))$. We also assume that the basic vectors satisfy the (UE),(LE) properties and (A3) condition. Then for μ-a.e. $\mathbf{x} \in M^q$,*

$$\dim_H(F) = \dim_P(F) = \overline{\dim}_B(F) = s_*,$$

where s_* is the unique solution of the functional equation:

$$I_s := \int_{M^q} \log\left\{\sum_{i=1}^q \phi_i(x_i)^{s_*}\right\} d\mu = 0, \text{ where } \mu = \mu_1 \times \mu_2 \cdots \times \mu_q. \tag{24}$$

For classical stochastic (and statistically self-similar) constructions, e.g. those described in [5], they instead consider the contraction ratios $|\Delta_{\omega * j}|/|\Delta_\omega| := C_j(\omega)$ as independent and identically distributed random variables. i.e. For each j, $\{C_j(\omega), \omega \in D_k\}$ are identically distributed and independent, although for fixed ω the set of random variables $\{C_j(\omega), j \leq n_{k+1}\}$ need not be independent. The corresponding Hausdorff dimension s satisfies the expectation equation $E(\sum_{j=1}^q C_j^s) = 1$, which is *not* equivalent to (24). The result of [5] is proved using a combination of martingale and potential theoretic methods. Consider inhomogeneous Moran set constructions, where the contraction ratios $C_j(\omega)$, $\omega \in D_k$ are governed by probability distributions that vary with step k. Then under suitable geometric constraints, see [21] the Hausdorff dimension of the corresponding limit set is given by $\dim_H(F) = s_*$, where

$$s_* = \liminf_{k\geq 1} s_k, \quad E\left(\sum_{(i_1,\ldots,i_k)} \prod_{j=1}^k (C_{i_j})^s\right) = 1.$$

Random symbolic constructions are also included in [17], and these include constructions with random vectors. They do not specifically generate the stochasticity using chaotic maps, and in their case they obtain only the inequality $\dim_H(F) \geq s$, where s satisfies the equation $\sum_{i=1}^q \exp\{s \int \log\phi_i \, d\mu_i\} = 1$. By a reverse Minkowski inequality this is consistent with the equality we obtain in (24).

Proof of Theorem 5: We first consider the case where $\inf_i \inf_{x_i} \phi_i(x_i) > 0$. Since the set F results from a MSC, conditions (A1)-(A3) hold and it is implicit that the ϕ_i are contractions. The corresponding contraction vector is given by $\tilde{\Xi}_k = (\phi_1(T_1^k(x_1)), \ldots, \phi_q(T_q^k(x_q))$. It suffices to compute the pre-dimensions s_k and calculate the limit $\liminf s_k$. We have:

$$\sum_{\omega\in D_k}\left(\prod_{j=1}^k \phi_{i_j}(T_{i_j}^j(x_{i_j}))\right)^{s_k} = 1. \tag{25}$$

A simple application of the binomial theorem implies that this expression is equivalent to:

$$\prod_{j=1}^k\left(\sum_{i=1}^q \{\phi_i(T_i^j(x_i)\}^{s_k}\right) = 1, \tag{26}$$

and so

$$\sum_{j=1}^k \log\left(\sum_{i=1}^q \{\phi_i(T_i^j(x_i)\}^{s_k}\right) = 0. \tag{27}$$

Now for *fixed* s, and by the ergodic theorem, we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^k \log\left(\sum_{i=1}^q \{\phi_i(T_i^j(x_i)\}^{s}\right) = \int_{M^q} \log\left\{\sum_{i=1}^q \phi_i(x_i)^s\right\} d\mu. \tag{28}$$

In the above, we have used the fact that the product system is ergodic. This is true provided each μ_i is mixing, [20]. Clearly, the value s_* which is the solution of (24) gives the right hand side of (28) as zero. By monotonicity of I_s, the value of s_* is unique. We now justify that $s_* = \liminf s_k$ by showing that for large k, $s_k = s_* + o(1)$. For finite (but large) k, we have

$$\sum_{j=1}^{k} \log \left(\sum_{i=1}^{q} \{\phi_i(T_i^j(x_i)\}^s \right) = k \left(\int_{M^q} \log \left\{ \sum_{i=1}^{q} \phi_i(x_i)^s \right\} d\mu + o(1) \right). \qquad (29)$$

By continuity of ϕ_i, it follows that $\forall\, \epsilon > 0$, the exists a K such that $\forall k \geq K$, we can choose s_k with $|s_* - s_k| < \delta$ and s_k satisfying (27). Hence $s_* = \liminf s_k$.

Suppose now that $\inf\limits_{1 \leq i \leq q} \{\inf_{x_i} \phi_i(x_i)\} = 0$, but $\int \phi_i \, d\mu_i \neq 0$. We now have $c_* = 0$, see equation (11). Therefore, we need to show that equation (13) applies for μ-typical orbits. If so, then Theorem 2 will establish the corresponding result. Proceeding, and using the notation of equation (13) we have that

$$\begin{aligned} d_k &= \min\nolimits_{1 \leq i \leq q} \{\phi_i(T_i^{k+1}(x_i))\}, \\ M_k &= \max\nolimits_{\omega} \left\{ \prod_{j=1}^{k} \phi_{i_j}(T_{i_j}^j(x_{i_j})) \right\}, \quad \omega = (i_1, \dots i_k). \end{aligned} \qquad (30)$$

We now show the following

Lemma 1 *Under the hypothesis of Theorem 5, we have for μ-a.e. $\mathbf{x} \in M^t$*

$$\lim_{k \to \infty} \frac{\log d_k}{\log M_k} = 0$$

Proof: We first notice that there is a constant $\lambda < 1$ such that

$$\begin{aligned} M_k &\leq (\sup_i \sup_{x_i} \phi_i(x_i))^k \leq \lambda^k, \\ \implies \log M_k &\leq k \log \lambda < 0, \end{aligned}$$

and

$$-\log d_k = \max\nolimits_{1 \leq i \leq q} \{-\log \phi_i(T_i^{k+1}(x_i))\} > 0.$$

Together these imply that

$$\frac{\log d_k}{\log M_k} \leq \frac{\max_{1 \leq i \leq q}\{-\log \phi_i(T_i^{k+1}(x_i))\}}{-k \log(\sup_i \sup_{x_i} \phi_i(x_i))} \leq \frac{\max_{1 \leq i \leq q}\{-\log \phi_i(T_i^{k+1}(x_i))\}}{-k \log \lambda}. \qquad (31)$$

We have to show that for μ_i-a.e. x_i, the right hand term of equation (31) goes to zero. We use a Borel-Cantelli argument as follows. Let

$$A_k^{(i)} = \{x_i \in M : \phi_i(T_i^k(x_i)) \leq \lambda^{\sqrt{k}}\}.$$

If $\mathbf{x} \in M^q$ is such that $x_i \notin A_k^{(i)}$ (for each component x_i), then $\phi_i(T_i^k(x_i)) < \lambda^{\sqrt{k}}$ and $\log d_k > \sqrt{k+1}\log\lambda$. By invariance of μ, and the fact that $\mu \in L^p$ we have by Hölder's inequality

$$\mu(A_k^{(i)}) \leq C(\mathrm{Leb}\{x \in M : \phi_i(x) < \lambda^{\sqrt{k}}\})^{\frac{(p-1)}{p}},$$

where C depends only on ϕ_i. Moreover, by Hölder continuity of ϕ_i, there is a constant $\gamma > 0$ such that $\mu(A_k^{(i)}) \leq C\lambda^{\gamma\sqrt{k}}$, and hence

$$\sum_{k=1}^{\infty} \mu_i(A_k^{(i)}) \leq C\sum_{k=1}^{\infty} \lambda^{\gamma\sqrt{k}} < \infty.$$

Therefore by the Borel-Cantelli Lemma that for μ-a.e. $\mathbf{x} \in M^q$,

$$\frac{\log d_k}{\log M_k} \leq \lambda^{\sqrt{k}}, \quad \text{(eventually as } k \to \infty). \tag{32}$$

□

We consider two further examples which can be easily generalized to other scenarios.

Example 1 Take $q = 2$, and suppose $\phi_1(x) := \phi(x)$ is non-constant with $\inf_{x\in M}\phi(x) = 0$. Suppose also that $\phi_2(x) = \lambda < 1/2$ (constant). In this example $M_k \geq \lambda^k$, and it need not be true that $\log d_k/\log M_k \to 0$. However in this example Theorem 3 applies, and the corresponding dimensions are given by equation (24).

Example 2 Consider the case where $\sup_{x\in M}\phi_i(x) = 1$ (for at least one $i \leq q$). Take $q = 2$, and for a given function $\phi(x) : M \to [0,1]$, and constant $\lambda < 1$ let $\phi_1(x) = \phi(x)$, $\phi_2(y) = \lambda(1-\phi(y))$. We will set $x = y$ and this dependency is to ensure that $\phi_1 + \phi_2 < 1$ for all steps of the construction. We also take $T_1 = T_2$. If $\int \log\phi\, d\mu < 0$ then an application of the ergodic theorem tells us that for μ a.e. x, the corresponding upper estimating vectors $\Psi_\omega^{(n)}$ are basic. If $\int \log\phi\, d\mu = 0$ then the upper-estimating vector need not be basic, and an explicit example would be to take $T(x)$ an interval map with a parabolic index greater than 1, i.e. a map of the form given in equation (21). If the upper-estimating vector is not basic then $\overline{\dim}_B(F)$ will depend on both the placement of the basic sets and the pre-dimension sequence s_k.

4.3 Stochastic Moran set constructions defined on subspaces of symbolic spaces

We consider a stochastic Moran set construction in the setting of Section 4.2 with $n_k = p$ fixed, and $A^{(k)}$ a fixed $q \times q$ matrix. Hence

$$Q_k = \{\omega \in D_k : A_{i_1 i_2}A_{i_2 i_3}\cdots A_{i_{k-1}i_k} = 1\}, \quad Q = \bigcup_k Q_k. \tag{33}$$

We consider the (dimension) properties of the limit set F defined by

$$F = \bigcap_{n\geq 1}\bigcup_{\omega\in Q_k}\Delta_\omega.$$

Our aim is to obtain a corresponding formula for the fractal dimension of F in terms of the limiting sequence s_k (defined using the full word sequence D_k), and the spectral properties of A. In the following we let $\rho(A)$ denote the spectral radius of a matrix A.

Theorem 6 *Suppose that $\{T_i, M, \mu_i\}_{i=1}^q$ form an ergodic system (i.e., each measure μ_i is ergodic w.r.t. T_i) and each $\phi_i : M \to [0,1]$ is positive Hölder continuous with $\int -\log \phi_i \, d\mu_i < \infty$. Suppose that the GMS, F arises from a MSC with a basic vector $\tilde{\Psi}$ generated via the vector sequence $\Xi_k = (\phi_1(T_1^k(x_1)), \ldots, \phi_q(T_q^k(x_q)))$, and fixed transition matrix $A^{(k)} = A$. We also assume that the basic vectors satisfy the (UE), (LE) properties and the $(A3)$ condition. Then for μ-a.e. $\mathbf{x} \in M^q$*

$$\dim_H(F), \dim_P(F), \overline{\dim}_B(F) \leq s_*.$$

Here s_* is the unique solution of the functional equation:

$$\int_{M^q} \log \left\{ \rho \left(A^T \Phi(\mathbf{x}, s_*) \right) \right\} d\mu = 0, \ \textit{where } \mu = \mu_1 \times \mu_2 \cdots \times \mu_q \tag{34}$$

and $\Phi(\mathbf{x}, s)$ is the diagonal matrix $\operatorname{diag}(\phi_1(x_1))^s, \ldots, \phi_q(x_q)^s),$.

Example 3 *Suppose admissible elements in Q are characterized by a $p \times p$ transition matrix A taking values in $\{0,1\}$, so that an element $\omega = (i_1, i_2, \ldots) \subset Q$ is admissible if $A_{i_j i_{j+1}} = 1$. Suppose we take contractions generated by a family of similarities with constant contraction rate α_i, $1 \leq i \leq q$ (and independent of the Moran construction step k). Then a straightforward calculation, see [17] implies that*

$$\dim_H(F) = \dim_B(F) = \dim_P(F) = \tilde{s}, \ \textit{with } \rho(A^T \operatorname{diag}(\alpha^{\tilde{s}_1}, \ldots, \alpha^{\tilde{s}_q}) = 1. \tag{35}$$

Remark 1 *Notice that we only obtain inequality in Theorem 6. If A is a constant matrix of 1s then we obtain Theorem 5 as before. If the spectral radius of A is equal to 1 then s^* is equal to zero, and hence the corresponding dimensions are zero.*

Remark 2 *The proofs would adapt easily to more general situations where $A^{(k)}$ varies with k. However, explicit bounds on the fractal dimensions in terms of the spectral properties of $A^{(k)}$ are perhaps less tractable.*

Remark 3 *If instead we have a homogeneous construction with stochastic vector $\Psi_\omega^{(k)} = \prod_{j=1}^k \phi(T^j(x))$. Then for μ-a.e. $x \in M$*

$$\dim_H(F), \, \dim_P(F), \, \overline{\dim}_B(F) = -\frac{\log \rho(A)}{\int_M \log \phi(x) \, d\mu}.$$

Proof: Following the proof of Theorem 5, the corresponding equation that replaces equation (26) is the following:

$$\rho \left(\prod_{i=1}^k A^T \Phi_i(\mathbf{x}, s_k) \right) = 1, \tag{36}$$

where $\Phi_i(\mathbf{x}, s) = (\text{diag}(\phi_1(T_1^i(x_1))^s, \ldots, \phi_p(T_p^i(x_p))^s)$. We can reduce this equation to the inequality $s_k \leq \tilde{s}_k$, where

$$\sum_{i=1}^{k} \rho\left(A^T \Phi_i(\mathbf{x}, \tilde{s}_k)\right) = 0. \tag{37}$$

This follows from $\rho(XY) \leq \rho(X)\rho(Y)$ (for matrices X, Y), and also from the fact that the left hand side of equation (36) is monotonically decreasing in s. We can now take limits in k as in the proof of Theorem 5 and hence obtain $\tilde{s}_* = \liminf_k s_k \leq s_*$, where s_* satisfies equation (34) as stated in the Theorem. In the case where $\inf \phi_i = 0$ equation (31) still applies for d_k and M_k when restricted to admissible words in Q. Hence, if the corresponding vectors satisfy properties (UE), (LE) and there exists a Gibbs measure satisfying (A3) then $dim_H(F), \dim_P(F), \overline{\dim}_B(F) \leq s_*$ as required.

5 Proof of main results

5.1 Proof of Theorem 1

The proof of Theorem 1 is given in several steps. We first obtain an estimate on how the measure m_Ψ scales on balls of radius r, as $r \to 0$. A second step is to show $\dim_H(F) = s_*$ using the existence of a basic vector sequence with the (LE), (UE) properties.

Lemma 2 *Suppose $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a* basic *sequence of vectors which satisfy the (LE) property and (A3). Take a GMS, F and $x \in F$. Then for any open ball $B(x, r)$, $(0 < r < 1)$ and any $\epsilon > 0$, there exists a Gibbs measure m_Ψ such that*

$$m_\Psi(B(x, r)) \leq C r^{s_* - \epsilon}. \tag{38}$$

The constant $C > 0$ is independent of r.

Proof: For $\omega \in D_k$ and for any $\beta < s_* := \liminf s_k$, by property (A3), we have

$$m_\Psi(\triangle_\omega) \leq L_1(\Psi_\omega^{(k)})^\beta. \tag{39}$$

Consider $x \in F$ and the ball $B(x, r)$ with $r \in (0, 1)$. Since $\overline{\Psi}$ is a (LE) vector, there exists an $M > 0$ such that the number $N(x, r)$ of $\triangle^{(j)}$ (in the Moran cover of F) with $\triangle^{(j)} \cap B(x, r) \neq \emptyset$ is bounded by M. Hence

$$m_\Psi(B(x, r)) \leq \sum_{j=1}^{N(x,r)} m_\Psi(\triangle^{(j)}) \leq \sum_{j=1}^{N(x,r)} L_1(\Psi_\omega^{(k)})^\beta, \tag{40}$$

where in the above summation ω corresponds to those for which $\triangle_\omega = \triangle^{(j)}$, and $\triangle^{(j)} \cap B(x, r) \neq \emptyset$. By (11) and using $N(x, r) \leq M$, we have

$$m_\Psi(B(x, r)) \leq L_1 M(\Psi_\omega^{(k)})^\beta \leq \frac{L_1 M}{c_*^\beta}(\Psi_\omega^{(k+1)})^\beta \leq \frac{L_1 M}{c_*^\beta} r^\beta. \tag{41}$$

This proves equation (38).

$\square$

Lemma 3 *Consider a MSC with a GMS, F. Suppose that (A3) holds, $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a* basic *sequence of vectors which satisfy the (UE), (LE) properties and $c_* > 0$. Then*

$$\dim_H F = \dim_H(m_\Psi) = s_*.$$

Proof: We first show that $\dim_H F \geq s_*$. Recall

$$\dim_H(m_\Psi) = \inf\{\dim_H(E) : \text{ with } m_\Psi(E) = 1\}.$$

Moreover, from the proof of Lemma 2, we have $m_\Psi(B(x,r)) \leq \frac{L_1 M}{c_*^\beta} r^\beta$, where $\beta < s_*$ is arbitrary. Hence it follows that $s_* \leq \dim_H(m_\Psi) \leq \dim_H F$, since β can be chosen arbitrarily close to s_*.

We now show that $\dim_H(F) \leq s_*$. Choose any $\beta > s_*$ then for Hausdorff measure $\mathcal{H}^\beta$ we have

$$\begin{aligned}\mathcal{H}^\beta(F) &\leq \liminf_{k\to\infty} \sum_{\omega\in D_k} \operatorname{diam}(\triangle_\omega)^\beta \leq \liminf_{k\to\infty} \sum_{\omega\in D_k} C(\Psi_\omega^{(k)})^\beta \\ &\leq \liminf_{k\to\infty} \sum_{\omega\in D_k} C(\Psi_\omega^{(k)})^{s_k} \leq CL_1 \left(\liminf_{k\to\infty} \sum_{\omega\in D_k} m_\Psi(\Delta_\omega) \right) < \infty,\end{aligned} \tag{42}$$

where in the second line we take the infimum along the subsequence s_k such that $s_k < \beta$ (which holds infinitely often). Thus $\mathcal{H}^\beta(F) \leq C$ and so $\dim_H F \leq \beta$. Since $\beta > s_*$ is arbitrary, it follows that $\dim_H(F) \leq s_*$. This completes the proof.

□

Lemma 4 *Consider a MSC with a GMS, F. Suppose that $\overline{\Psi} = \{\overline{\Psi}^{(k)}\}$ is a basic collection of vectors which satisfies the (UE), (LE) properties and $c_* > 0$. Then*

$$\dim_P F \leq \overline{\dim}_B F \leq s^*.$$

Proof: We extend the ideas used in [17, Page 141]. Suppose (by contradiction) that $\overline{\dim}_B(F) > s^*$. Given the sequence s_k and the fact $s^* = \limsup_{k\to\infty} s_k$, then for all $\delta > 0$, there exists $\tilde{k} > 0$ such that such that $\forall\, k \geq \tilde{k}$, $\overline{\dim}_B(F) - 3\delta > s_k$. By definition of the pre-dimension sequence s_k and noting that and corresponding sequence of pressure functions $P_k : s \mapsto P_k(s \log \Phi^{(k)})$ are decreasing in s, we have for all $k \geq \tilde{k}$

$$P_k((\overline{\dim}_B(F) - 3\delta) \log \Phi^{(k)}) < 0. \tag{43}$$

Let $\beta = \overline{\dim}_B F$, then by the definition of upper box dimension we have

$$\limsup_{\epsilon\to 0} \frac{\log N_\epsilon(F)}{-\log \epsilon} = \beta.$$

Hence given $\delta > 0$, there is a sequence $\epsilon_n = \epsilon_n(\delta) \to 0$, $(n \to \infty)$, such that

$$N_\epsilon(F) \geq \epsilon_n^{\delta-\beta}. \tag{44}$$

Given $\delta > 0$, let ϵ be a representative from the sequence ϵ_n, which can be made arbitrarily small. Let $\{\Delta^j\}, j = 1, \cdots, N^\epsilon(F)$ be the Moran covering of F at this ϵ-scale. We have $N^\epsilon(F) \geq N_\epsilon(F)$. Since $0 < d < 1$ there exists $A > 0$ such that for $j = 1, \cdots, N^\epsilon(F)$:

$$\frac{\epsilon}{A} \leq \Psi_\omega^{(n(\omega))} \leq \epsilon. \tag{45}$$

Hence there exist uniform constants C_1 and C_2 such that

$$C_1 \log(\frac{1}{\epsilon}) \leq n(x_j) \leq C_2 \log(\frac{A}{\epsilon}).$$

In the Moran covering the $n(\omega)$ can take on at most $C_3 := C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon}) > 0$ possible values. By the pigeon hole principle there exists a positive integer $\alpha := \alpha(\delta)$ with $C_1 \log(\frac{1}{\epsilon}) \leq \alpha \leq C_2 \log(\frac{A}{\epsilon})$ such that for a sufficient small ϵ,

$$\sharp\{\omega \text{ such that } n(\omega) = \alpha\} \geq \frac{N^\epsilon(F)}{C_3} \geq \frac{N_\epsilon(F)}{C_3} \geq \frac{\epsilon^{\delta-\beta}}{C_3} \geq \epsilon^{2\delta-\beta}. \tag{46}$$

Recall that for any fixed number s, the potential Φ, given by the function $\Phi(x) := s \log \Psi_{\omega_1}(\mathbf{w})$, where $\mathbf{w} := (\omega_1, \omega_2, \ldots) \in [D_k]$ is only dependent on the first coordinate ω_1. Therefore,

$$(S_n\Phi)(x) = \sum_{j=1}^{n} \Phi(\sigma_k^j x) = s \log \prod_{j=1}^{n} \Psi_{\omega_j}^{(k)},$$

and hence $\exp(S_n\Phi)(x) = (\prod_{j=1}^n \Psi_{\omega_j}^{(k)})^s$. We have

$$P_k(\Phi(x)) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(\omega_1,\ldots,\omega_n)} \inf_{x\in\Delta_{(\omega_1,\ldots,\omega_n)}} \exp\left(\sum_{j=0}^{n-1} s \log([\sigma_k^j \mathbf{w}(x)]^{(1)})\right), \tag{47}$$

and so

$$P_k(\Phi(x)) = \log\left(\sum_{\omega\in D_k} (\Psi_\omega^{(k)})^s\right).$$

If we put $k = \alpha$ and apply equation (46) with $s = \overline{\dim}_B(F) - 3\delta = \beta - 3\delta$, we obtain

$$\begin{aligned} P_\alpha((\beta - 3\delta)\Psi_{\omega_1}^{(\alpha)}) &\geq \log\left(\sum_{\omega\in\{\Delta^{(j)}\}} (\Psi_\omega^{(\alpha)})^{\beta-3\delta}\right) \\ &\geq \log\left(\left(\frac{\epsilon}{A}\right)^{\beta-3\delta} \epsilon^{2\delta-\beta}\right) \\ &= \log(\epsilon^{-\delta} A^{3\delta-\beta}) \geq 0. \end{aligned} \tag{48}$$

where $\{\Delta^{(j)}\}$ is the corresponding Moran cover. In the final inequality we have used the fact that A is independent of δ. Hence $\beta - 3\delta < s_\alpha$. The constant α depends on the sequence ϵ_n, and can be taken arbitrarily large. This implies that there is a subsequence $k_j \to \infty$, such that $\beta - 3\delta < s_{k_j}$, and hence $\beta - 3\delta < \limsup s_k = s^*$

for every $\delta > 0$. The sequence k_j implicitly depends on δ, but for each $\delta > 0$ such an (infinite) sequence will always exist. Hence $\beta \leq s^*$ in contradiction to (43).

$\square$

We now provide a lower bound for the box/packing dimensions when the construction is conformal.

Lemma 5 *Suppose a GMS, F arising from a MSC satisfies the conformal condition as in Definition 8, then*

$$\overline{\dim}_B F = s^*.$$

Proof: Using Lemma 4, it suffices to prove the lower bound. For each $\beta < s^*$, there exists a subsequence $\{s_k\}$ such that for each k, $\beta < s_k$. Moreover, the conformal condition implies that $B(x, C^{-1}\Psi_\omega^{(k)}) \subseteq \triangle_\omega$, for each $\omega \in D_k$. Using an equivalent definition of box dimension, see Section 6, we let

$$W^s(F) := \lim_{r \to 0} \sup\{\sum_i \mathrm{diam}(B_i)^s : \mathrm{diam}(B_i) \leq r, B_i^\circ \cap B_j^\circ = \emptyset (i \neq j), B_i \cap F \neq \emptyset\}.$$

Then

$$\overline{\dim}_B F := \sup\{s : W^s(F) = \infty\} = \inf\{s : W^s(F) = 0\},$$

and hence,

$$\begin{aligned} W^\beta(F) &\geq \limsup_{k \to \infty} \sum_{\omega \in D_k} \mathrm{diam}(B(x, C^{-1}\Psi_\omega^{(k)}))^\beta \\ &\geq \limsup_{k \to \infty} \sum_{\omega \in D_k} C^{-1}(\Psi_\omega^{(k)})^{s_k} \\ &\geq \limsup_{k \to \infty} \sum_{\omega \in D_k} C^{-1} L_1^{-1} m_\Psi(\triangle_\omega) > 0, \end{aligned}$$

which implies that $\overline{\dim}_B F > \beta$. Since β is arbitrary, it follows that $\overline{\dim}_B = s^*$.

$\square$

Lemma 6 *Suppose a GMS, F satisfies the conformal condition, then* $\dim_P F = \overline{dim_B} F$.

Proof: By Lemma 8 in Section 6, it suffices to show that for any open set V, $\overline{\dim}_B(F \cap V) \leq \overline{\dim}_B F = s^*$, provided $F \cap V \neq \emptyset$. We do this as follows. Clearly $\overline{\dim}_B(F \cap V) \leq \overline{\dim}_B F$. Moreover, for any open set V with $F \cap V \neq \emptyset$, there exists $\tilde{\omega} \in D_N$ such that $\triangle_{\tilde{\omega}} \subset V$. Taking $\beta < s^*$, there exists a subsequence s_k with $\beta < s_k$, such that

$$\begin{aligned} W^\beta(F \cap V) &\geq \limsup_{k \to \infty} \sum_{\omega \in D_k, \triangle_\omega \subseteq \triangle_{\tilde{\omega}}} \mathrm{diam}(B(x, C^{-1}\Psi^{(k)}))^\beta \\ &\geq \limsup_{k \to \infty} \sum_{\omega \in D_k, \triangle_\omega \subseteq \triangle_{\tilde{\omega}}} C^{-1}(\Psi^{(k)})^{s_k} \\ &\geq \limsup_{k \to \infty} \sum_{\omega \in D_k, \triangle_\omega \subseteq \triangle_{\tilde{\omega}}} C^{-1} L_1^{-1} m_\Psi(\triangle_\omega) \\ &= C^{-1} L_1^{-1} m_\Psi(\triangle_{\tilde{\omega}}) > 0. \end{aligned}$$

Hence we obtain $s^* = \overline{\dim}_B(F \cap V) = \overline{\dim}_B(F)$, which completes the proof.

$\square$

5.2 Proof of Theorem 2

The proof of Theorem 2 is as follows. We claim first of all that $\dim_H F \leq s_*$. The proof of this claim follows step by step the proof of Lemma 3 via equation (42). Hence it suffices to show only that $\dim_H(F) \geq s_*$.

Suppose $\beta < s_*$, then there exists a $K \in \mathbb{N}$ such that for all $k \geq K$, $s_k > \beta$. Moreover for any $\omega \in D$, $(\Psi_\omega^{(k)})^{s_k} < (\Psi_\omega^{(k)})^\beta$.

Since $\log d_k / \log M_k \to 0$, there exists an $\epsilon > 0$ such that for all $k \geq K$, $M_k^{\epsilon/2} < d_k^{\beta+\epsilon}$ and hence that $M_k^{\epsilon/2}/d_k^{\beta+\epsilon} < 1$.

Now take the Moran cover $\triangle^{(j)}$ such that $m_\Psi(\triangle^{(j)}) \leq L_1(\Psi_\omega^{(k)})^{s_k} < (\Psi_\omega^{(k)})^\beta$. Given $r > 0$, the Moran cover $\triangle^{(j)}$ has the property that $(\Psi_\omega^{(n(\omega))}) \geq r$ and $(\Psi_\omega^{(n(\omega)+1)}) < r$. Since $s_k > \beta$ we choose ϵ sufficiently small so that $s_k > \beta + \epsilon$. Therefore, we obtain the following series of estimates:

$$\begin{aligned}
m_\Psi(B(x,r)) &\leq \sum_{j=1}^{N(x,r)} m_\Psi(\triangle^{(j)}) \leq \sum_{j=1}^{N(x,r)} L_1(\Psi_\omega^{(k(\omega))})^{\beta+\epsilon} \\
&\leq \sum_{j=1}^{N(x,r)} \frac{L_1}{d_{k(\omega)}^{\beta+\epsilon}}(\Psi_\omega^{(k(\omega)+1)})^{\beta+\epsilon} \leq \sum_{j=1}^{N(x,r)} \frac{L_1 M^{\epsilon/2}}{d_{k(\omega)}^{\beta+\epsilon}}(\Psi_\omega^{(k(\omega)+1)})^{\beta+\frac{\epsilon}{2}} \\
&\leq L_1 M r^{\beta+\frac{\epsilon}{2}}.
\end{aligned} \tag{49}$$

It follows that $\dim_H(m_\Psi) \geq s_*$, since we can choose β arbitrarily close to s_* and ϵ arbitrarily close to 0. It follows that $\dim_H(F) \geq \dim_H(m_\Psi) \geq s_*$ and hence we obtain $\dim_H(F) = s_*$

We now turn to the box dimension. First consider the case where the vector $\overline{\Psi}$ is upper-estimating, but the construction is not conformal. We can repeat the proof of Lemma 4, but we note that the constant A appearing in equation (45) is now dependent on ϵ. Taking again the Moran covering $\{\Delta_\omega^j\}, j = 1, \cdots, N^\epsilon(F)$ of F at scale ϵ, we have $\Psi_\omega^{(n(\omega)+1)} < r \leq \Psi_\omega^{(n(\omega))}$. Recalling that $d_k = \min_{1 \leq j \leq n_{k+1}} c_j^{(k+1)}$, we obtain

$$\epsilon > \Psi_\omega^{(n(\omega)+1)} = \Psi_\omega^{(n(\omega))} c_j^{(n+1)}(\omega) \geq d_{n(\omega)+1} \Psi_\omega^{(n(\omega))},$$

and hence $\Psi_\omega^{(n(\omega))} \leq \epsilon d_{n(\omega)+1}^{-1}$. Since $\lim_{k\to\infty} \frac{\log d_k}{\log M_k} = 0$ it follows that for all $\eta > 0$, there exists a K, such that $\forall\, k \geq K$, $1 > d_k > M_k^\eta > 0$, and therefore

$$\Psi_\omega^{(n(\omega))} \leq \epsilon M_k^{-\eta}.$$

Hence by definition of M_k we obtain (for arbitrary $\eta > 0$): $\epsilon < \Psi_\omega^{n(\omega)} \leq \epsilon^{\frac{1}{1+\eta}}$. Following step by step the proof of Lemma 4, we obtain $\overline{\dim}_B(F) \leq s^*$.

When the construction is conformal, the upper bound for $\dim_P F$ and $\overline{\dim}_B F$ is obtained as in the calculation directly above. The lower bounds follow from Lemmas 5 and 6.

$\square$

5.3 Proof of Theorem 3

To prove Theorem 3 we consider a truncated construction, such as that considered in [8]. We remove words $\omega \in D_k$ for which $c_j^{(k)} < \epsilon$, and define

$$D_k(\epsilon) = \{\omega \in D_k : c_j^{(k)} \geq \epsilon, \forall i \leq k-1\}, \quad \tilde{D}_k(\epsilon) = \{\triangle_\omega : \omega \in D_k(\epsilon)\},$$

and

$$E_k(\epsilon) = \bigcup_{\omega \in D_k(\epsilon)} \triangle_\omega, \qquad F(\epsilon) = \bigcap_{k \geq 0} E_k(\epsilon).$$

For the ϵ-truncated construction $F(\epsilon)$ of F, the associated vectors $\{\Psi_\omega^{(k)}\}$ are both upper and lower-estimating, and so we can use Theorem 1 to find the fractal dimension of $F(\epsilon)$. In particular the dimension of $F(\epsilon)$ can be found by taking appropriate limits along the pre-dimension sequences $s_k(\epsilon)$, where $s = s_k(\epsilon)$ solves the equation $P_k(s\log(\Psi_\omega^{(k)}\mathcal{X}_{D_k(\epsilon)}(\omega))) = 0$. Here $\mathcal{X}_{D_k(\epsilon)}(\omega)$ denotes the indicator function of $D_k(\epsilon)$. The following lemma makes explicit the relation between $s_k(\epsilon)$ and s_k, the latter value being the solution to $P_k(s\log(\Psi^{(k)})) = 0$.

Lemma 7 *Suppose s_k and $s_k(\epsilon)$ are solutions to the respective pressure equations*

$$P_k(s\log(\Psi_\omega^{(k)})) = 0, \quad P_k(s\log(\Psi_\omega^{(k)}\mathcal{X}_{D_k(\epsilon)}(\omega))) = 0.$$

Suppose that $s_* = \liminf s_k > 0$. Then for k sufficiently large,

$$0 \leq s_k - s_k(\epsilon) \leq \mathcal{O}(\epsilon^{s*/2}), \tag{50}$$

where the implied constant in $\mathcal{O}(\cdot)$ is independent of k.

Proof: The arguments follow close to [8] and we provide the main steps. Suppose that $s_* > 0$. Firstly, there exist constants $0 < \alpha, \beta < 1$ such that

$$\alpha < \inf_k \max_{1 \leq j \leq n_k} c_j^{(k)}, \quad \sup_k \max_{1 \leq j \leq n_k} c_j^{(k)} < \beta.$$

Observe that for any $\omega \in D_k$ (and $i \leq k$) we have that

$$\sum_{j=1}^{n_i} (c_j^{(i)})^{s_k} \geq \alpha^{s_k} \geq \alpha^{\tilde{d}}, \tag{51}$$

$$\sum_{j=1}^{n_i} (c_j^{(i)})^{s_k} \mathcal{X}_{D_k(\epsilon)}(\omega) > \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k} - M\epsilon^{s_k}. \tag{52}$$

where $\tilde{d}$ is the dimension of the space. From equation (52) we obtain

$$\sum_{j=1}^{n_i} (c_j^{(i)})^{s_k} \mathcal{X}_{D_k(\epsilon)}(\omega) > \left(1 - \frac{M\epsilon^{s_k}}{\alpha^{\tilde{d}}}\right) \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k}. \tag{53}$$

Now, for any $\gamma > 0$ we have

$$\sum_{j=1}^{n_i} (c_j^{(i)})^{s_k} \mathcal{X}_{D_k(\epsilon)}(\omega) = \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k-\gamma} (c_j^{(k)})^{\gamma} \mathcal{X}_{D_k(\epsilon)}(\omega) \leq \beta^{\gamma} \sum_{j=1}^{n_i} (c_i^{(k)})^{s_k-\gamma} \mathcal{X}_{D_k(\epsilon)}(\omega). \quad (54)$$

Taking products and combining equations (53), (54) we obtain

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k-\gamma} \mathcal{X}_{D_k(\epsilon)}(\omega) \geq \beta^{-k\gamma} \left(1 - \frac{M\epsilon^{s_k}}{\alpha^{\tilde{d}}}\right)^k \prod_{i=1}^{k} \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k}. \quad (55)$$

We observe that s_k and $s_k(\epsilon)$ satisfy the pre-dimension equations

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k(\epsilon)} \mathcal{X}_{D_k(\epsilon)}(\omega) = 1, \quad \prod_{i=1}^{k} \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k} = 1. \quad (56)$$

Since $s_* > 0$, there exists k_0, such that $s_k > s_*/2$ for all $k \geq k_0$. Moreover, there exists ϵ_0, such that for all $\epsilon < \epsilon_0$, we have

$$0 < \gamma_\epsilon := \log(1 - M\alpha^{-\tilde{d}}\epsilon^{s_k})/(\log\beta) < \log(1 - M\alpha^{-\tilde{d}}\epsilon^{\frac{s_*}{2}})/(\log\beta) < \frac{s_*}{2}. \quad (57)$$

From equations (55) and (55), we see that for any $\gamma < \gamma_\epsilon$ we have

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i} (c_j^{(i)})^{s_k-\gamma} \mathcal{X}_{D_k(\epsilon)}(\omega) \geq 1, \quad (58)$$

and therefore have $s_k(\epsilon) > s_k - \gamma$. From the observation that $s_k \geq s_k(\epsilon)$ the result now follows.

□

We now claim that $\dim_H(F) = \liminf s_k = s_*$. Since $F(\epsilon) \subset F$, we have $\dim_H(F) \geq s_*(\epsilon)$, and by Lemma 7, we have $\lim_{\epsilon\to 0} s_*(\epsilon) = s_*$. Hence $\dim_H(F) \geq s_*$. For the upper bound we just apply the same argument as Lemma 3.

For the upper-box and packing dimensions we claim that $\overline{\dim}_B(F) = \limsup s_k = s^*$. For a monotonically decreasing sequence $\epsilon_n \to 0$ let $F^* = \bigcup_{n=1}^{\infty} F(\epsilon_n)$. Then by the closure property of upper-box dimension we have $\overline{F^*} = F$, and so $\overline{\dim}_B(F^*) = \overline{\dim}_B(F)$. It therefore suffices to calculate $\overline{\dim}_B(F^*)$. By Theorem 1, and $\forall\ \epsilon > 0$, we have $\overline{\dim}_B F(\epsilon) = \dim_P(F(\epsilon))$. These dimensions equal $\limsup_k s_k(\epsilon)$. Furthermore, $\dim_P(F^*) = \limsup_{n\to\infty} \dim_P(F(\epsilon_n)) = s^*$. This completes the proof.

6 Appendix: Background on fractal dimension and its computation

6.1 Hausdorff, Box and Packing dimensions

In this section we give the relevant background on dimension theory, see [4, 5, 11] for a more general discussion.

Suppose F is a non-empty subset in $\mathbb{R}^d$. For any non-negative number s and $\epsilon > 0$, let

$$\mathcal{H}^s_\epsilon(F) := \inf\left(\sum_i (\mathrm{diam}(U_i))^s\right), \tag{59}$$

where the infimum is taken over all covers $\{U_i\}$ with $\mathrm{diam}(U_i) < \epsilon$. As ϵ decreases, the class of permissible covers of F in (59) is reduced, and therefore, the infimum $\mathcal{H}^s_\epsilon$ increases. The limit $\mathcal{H}^s(F) := \lim_{\epsilon\to 0}\mathcal{H}^s_\epsilon(F)$ exists, and is called as *Hausdorff measure*. The corresponding *Hausdorff dimension* of F is defined by

$$\dim_H(F) := \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}. \tag{60}$$

A disadvantage of Hausdorff dimension lies in its calculation. Alternative definitions of dimension, which are perhaps easier to estimate are the following.

The (upper) box dimension is relatively easier to estimate than Hausdorff dimension, and is defined as follows. Given a non-empty set $F \subset \mathbb{R}^d$, and $\epsilon > 0$, let $N(\epsilon)$ denote the smallest number of ϵ-balls needed to cover F. The (upper) box dimension of F is defined by:

$$\overline{\dim}_B(F) := \limsup_{\epsilon\to 0}\frac{\log N(\epsilon)}{\log(1/\epsilon)}. \tag{61}$$

Analogous to Hausdorff dimension, there is an alternative description of upper box dimension [16]: for any non-negative number s, let

$$W^s(F) := \lim_{\epsilon\to 0}\sup\{\textstyle\sum_i \mathrm{diam}(B_i)^s : \begin{array}{l} B_i \text{ is a ball with } \mathrm{diam}(B_i) \leq \epsilon, \\ B_i^\circ \cap B_j^\circ = \emptyset (i \neq j), B_i \cap F \neq \emptyset\}, \end{array} \tag{62}$$

then

$$\overline{\dim_B}(F) := \sup\{s : W^s(F) = \infty\} = \inf\{s : W^s(F) = 0\}. \tag{63}$$

It is worth mentioning that $W^s(\cdot)$ in equation (62) usually does not define a measure (due to lack of subadditivity). Moreover $\overline{\dim_B}(F) = \overline{\dim_B}(\overline{F})$, where $\overline{F}$ is the closure of F. Hence box dimension can give positive values to countable sets (unlike Hausdorff dimension).

Comparing equation (59) with equation (62), we have

$$\dim_H(F) \leq \overline{\dim}_B(F). \tag{64}$$

We now introduce packing dimension and packing measure. Let

$$\mathcal{P}^s_\epsilon(F) := \sup\{\sum_i \mathrm{diam}(B_i)^s\}$$

where the supremum is taken over a collection of disjoint balls $\{B_i\}$ of radius at most ϵ and with centers in F. The limit $\mathcal{P}^s_0(F) := \lim_{\epsilon\to 0}\mathcal{P}^s_\epsilon(F)$ exists. However, by considering countable dense sets, it is easy to see that $\mathcal{P}^s_0$ is not a measure (again, due to lack of subadditivity). Hence, we modify $\mathcal{P}^s_0$ to

$$\mathcal{P}^s(F) := \inf\left\{\sum_i \mathcal{P}^s_0(F_i) : F \subset \bigcup_{i=1}^\infty F_i\right\}, \tag{65}$$

which is a measure, and is called an s-dimensional packing measure. The *packing dimension* is naturally defined as

$$\dim_P(F) := \sup\{s : \mathcal{P}^s(F) = \infty\} = \inf\{s : \mathcal{P}^s(F) = 0\}. \tag{66}$$

For a general set $F \subset \mathbb{R}^d$, the following relations hold:

$$\dim_H(F) \leq \dim_P(F) \leq \overline{\dim}_B(F), \quad \text{and} \quad \mathcal{H}^s(F) \leq \mathcal{P}^s(F). \tag{67}$$

Suitable examples show that none of inequalities in (67) can be replaced by equalities [5].

The following lemma is useful for studying packing and box dimension, especially for fractal sets with some degree of self similarity.

Lemma 8 *[4, Corollary 3.9] Let $F \subset \mathbb{R}^n$ be compact and for all open sets V that intersect with F suppose that $\overline{\dim}_B(F \cap V) = \overline{\dim}_B(F)$. Then $\dim_P(F) = \overline{\dim}_B(F)$.*

6.2 Relation between conformality and the (LE) property

The following lemma gives the relationship between a conformal construction and a construction which admits a lower estimating vector.

Lemma 9 *If a vector $\overline{\Psi}$ is conformal, and $c_* > 0$, then the vector $\overline{\Psi}$ satisfies (LE) property.*

Proof: For any fixed $0 < r < 1$, and any $x \in F$, consider the open ball $B(x, r)$ centered in x with radius of r, and let $N(x, r)$ be the number of Moran covering $\{\triangle^{(j)}\}$ that have nonempty intersection with $B(x, r)$. Hence

$$B(x, r) \bigcup \left(\cup_{j=1}^{N(x,r)} \triangle^{(j)} \right) \subseteq B(x, R),$$

where

$$R = 2r + \sup_j \operatorname{diam}(\triangle^{(j)}).$$

Using the conformal condition and recalling from the definition of $\triangle^{(j)}$, we can choose elements $\triangle_\omega$, $\omega \in D_{n+1}$ such that $\Psi_\omega^{(n)} \geq r$ and $\Psi_\omega^{(n+1)} \leq r$. Therefore,

$$R \leq 2r + \sup_j C\Psi_{\omega_j}^{(k)} \leq 2r + \frac{r}{c_*},$$

and

$$\operatorname{diam}(\triangle^{(j)}) \geq C^{-1}\Psi_{\omega_j}^{(k)} \geq C^{-1}r.$$

Hence it follows that for each $x \in F$ and $0 < r < 1$

$$N(x, r) \leq \frac{2r + c_*^{-1}r}{C^{-1}r} = \frac{2 + c_*^{-1}}{C^{-1}} < \infty.$$

Therefore the vector $\overline{\Psi}$ satisfies (LE) property.

□

6.3 Background on thermodynamic formalism

For inhomogeneous Moran set constructions we used intermediate constructions based on finite symbolic schemes to calculate the fractal dimension. We review relevant background on thermodynamic formalism for these finite symbolic schemes, see for example [4, 15, 17].

Consider the finite symbolic dynamical system $(\sum_p^+, \sigma)$, where $\sum_p^+ = \{0, \cdots, p-1\}^{\mathbb{N}}$ and $\sigma : \sum_p^+ \to \sum_p^+$ as the left-shift map. Suppose $Q \subset \Sigma_p^+$ is a σ-invariant set. If $\omega \in Q$, then we write $\omega = (i_1, i_2, \ldots)$, with $i_j \in \{0, \ldots, p\}$ an admissible sequence (for $j \geq 1$). We turn Σ^+ into a metric space using a standard symbolic metric, such as that given in Section 2. Given $\omega \in Q$, we write $C_{i_1,\ldots i_k}(\omega) \subset Q$ as the k-length cylinder set that contains ω. Given an $\alpha-$Hölder continuous function $\phi : Q \to \mathbb{R}^+$, let $S_k(\phi) := \sum_{i=0}^{k-1} \phi \circ \sigma^i$, then the *(topological) pressure* $P(\phi)$ is defined by

$$P(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\substack{(i_1,\ldots,i_n) \\ \text{admissible}}} \inf_{\omega \in C_{i_1,\ldots,i_n}} \exp(S_n(\phi)(\omega)) \right). \tag{68}$$

For topological dynamical systems, the following variational principle holds. Let $\mathcal{M}(Q)$ denote the space of σ-invariant measures on Q. Then for $\phi : Q \to \mathbb{R}^+$ Hölder continuous we have

$$P(\phi) = \sup_{\mu \in \mathcal{M}(\mathcal{Q})} \left(h_\mu(\sigma) + \int_Q \phi \, d\mu \right),$$

where $h_\mu(\sigma)$ is the topological entropy of σ. The measure $\mu = \mu_\phi$ that gives rise to the supremum is called an *equilibrium measure*. This measure always exists, but need not be unique. Another measure of significance is that of a *Gibbs measure*. For any $\alpha-$Hölder continuous map $\phi : \sum_p^+ \to \mathbb{R}^+$, an invariant measure μ is called a Gibbs measure for the potential ϕ if there exists a constant $D > 1$ such that

$$D^{-1} \leq \frac{\mu\{y : y_i = x_i, i = 1, \cdots, n\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k(x)))} < D \tag{69}$$

for all $x = (x_1, x_2, \cdots) \in \sum_p^+$ and $n \geq 0$. In fact, for the shift map σ on a finite symbolic space, the hypothesis of the $\alpha-$Hölder continuity of the potential ϕ ensures the existence and uniqueness of the Gibbs measure and its coincidence with the equilibrium state for ϕ. However for more general symbolic schemes less is known about the existence of such measures. To study results on fractal dimension, the potential ϕ of interest is that which depends only on the first coordinate, i.e., $\phi(x) = \phi(x_1)$. In [15] it shown that for given numbers $0 < \lambda_i < 1, i = 1, \cdots, p$, and potential function $\phi : \sum_p^+ \to \sum_p^+$ defined by $\phi(x) = \phi(x_1, x_2, \cdots) = \log \lambda_{x_1}^{-1}$, the equation $P(s\phi) = 0$ has a unique solution in s. Moreover ϕ is Hölder continuous. This unique solution s is equal to the Hausdorff, Packing and Boxing dimensions of certain repelling invariant sets generated by IFS, see for example [4, 16].

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